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Plane Symmetric Vacuum Bianchi Type-III Cosmological Model in Brans-Dicke Theory

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Abstract We have obtained an exact solution of the vacuum Brans-Dicke (Phys. Rev. 124:925, 1961) field equations for the metric tensor of a spatially homogeneous and anisotropic model. Some physical properties of the model are also studied.

Keywords Brans Dicke theory · Plane symmetric Bianchi type-III model

1 Introduction

In the recent years there has been a lot of interest in the study of large scale structure of the universe because of the fact that the origin of structure in the universe is one of the greatest cosmological mysteries even today. Dehnen and Obregon [1, 3] obtained exact solutions of vacuum Brans-Dicke cosmological equations and proved that it has no analogy in Einstein's theory even for the larger values of the parameter. Van den Bergh [4] obtained a complete set of solutions of vacuum Brans-Dicke equations for a static spherically metric and proved that classical general relativity is more Machian than Brans-Dicke theory it self. Shri Ram [5] obtained an exact solution of the vacuum Brans-Dicke field equations for a spatially homogeneous and anisotrophic Bianchi Type-I configuration and shown the well known Kasner Universe as a special case. A plane symmetric vacuum Bianchi Type-I cosmological model in Brans-Dicke scalar-tensor theory of gravitation has been studied by same author [6]. An exact solution of vacuum Brans-Dicke equations for the metric tensor of Bianchi type-VI configuration has been obtained by Shri Ram et al. [7].

In this paper, we have obtained an exact solution of the vacuum field equations of Brans-Dicke theory for the metric tensor of a spatially homogeneous and anisotropic plane symmetric Bianchi type-III cosmological model.

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2 The Field Equations and Their Solutions

The spatially homogeneous and anisotrophic plane symmetric Bianchi type-III line element is

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{-2\alpha x}dy^2 - C^2dz^2, \tag{1}$$

where α is constant. A, B, C are functions of time 't' which measures the expansion or contraction of the model in the transverse and longitudinal direction. We number the coordinates x, y, z, t as 1, 2, 3, 4 respectively.

The Brans-Dicke equations in vacuum [4] are

$$R_{ij} = -\frac{\omega}{\phi^2}\phi_{;i}\phi_{;j} - \frac{1}{\phi}\phi_{;i}\phi_{;j} \tag{2}$$

and

$$\Box\phi = 0, \tag{3}$$

where the Brans-Dicke scalar ϕ is the reciprocal of the gravitational constant. Other symbols have their usual meaning.

Applying (2) and (3) to the metric (1), we get

$$\frac{A_{44}}{A} - \frac{\alpha^2}{A^2} + \frac{A_4}{A}\frac{B_4}{B} + \frac{A_4}{A}\frac{C_4}{C} = -\frac{A_4}{A}\frac{\phi_4}{\phi}, \tag{4}$$

$$\frac{B_{44}}{B} + \frac{A_4}{A}\frac{B_4}{B} + \frac{\alpha^2}{A^2} + \frac{B_4}{B}\frac{C_4}{C} = -\frac{B_4}{B}\frac{\phi_4}{\phi}, \tag{5}$$

$$\frac{C_{44}}{C} + \frac{A_4}{A}\frac{C_4}{C} + \frac{B_4}{B}\frac{C_4}{C} = -\frac{C_4}{C}\frac{\phi_4}{\phi}, \tag{6}$$

$$\frac{A_{44}}{A} + \frac{B_{44}}{B} + \frac{C_{44}}{C} = -\frac{\omega}{\phi^2}\phi_4^2 - \frac{\phi_{44}}{\phi}, \tag{7}$$

$$-\alpha\frac{B_4}{B} + \alpha\frac{A_4}{A} = 0, \tag{8}$$

$$\frac{\phi_{44}}{\phi} + \left(\frac{B_4}{B} + \frac{A_4}{A} + \frac{C_4}{C}\right)\frac{\phi_4}{\phi} = 0. \tag{9}$$

On solving (8) we get

$$B = AK,$$

where K is constant of integration.

Without loss of generality we take $K = 1$

$$\therefore B = A.$$

Now the field equations (4)–(9) reduces to

$$\frac{A_{44}}{A} + \left(\frac{A_4}{A}\right)^2 + \left(\frac{A_4}{A}\right)\left(\frac{C_4}{C}\right) = -\frac{A_4}{A}\frac{\phi_4}{\phi}, \tag{10}$$

$$\frac{C_{44}}{C} + 2\frac{A_4}{A}\frac{C_4}{C} = -\frac{C_4}{C}\frac{\phi_4}{\phi}, \tag{11}$$

$$2\frac{A_{44}}{A}+\frac{C_{44}}{C}=-\omega\left(\frac{\phi_4}{\phi}\right)^2-\frac{\phi_{44}}{\phi}, \tag{12}$$

$$\frac{\phi_{44}}{\phi}+\left(2\frac{A_4}{A}+\frac{C_4}{C}\right)\frac{\phi_4}{\phi}=0, \tag{13}$$

where the suffix 4 denotes ordinary differentiation with respect to 't'.

In order to solve these equations we take u and v the logarithmic derivative $\frac{\phi_4}{\phi}$ and $\frac{A_4}{A}$ of ϕ and A respectively. Then (10) and (13) readily gives

$$\frac{C_4}{C}=-\left(u+2v+\frac{v_4}{v}\right), \tag{14}$$

$$\frac{C_4}{C}=-\left(u+2v+\frac{u_4}{u}\right). \tag{15}$$

Solving (14) and (15) we get $u=vk$ with k is a constant of integration.

From (11), (12) and (14) we obtain

$$v_4+\frac{(2k^2+4k+\omega k^2+6)}{2k+4}v^2=0, \tag{16}$$

which on integration gives

$$v=(at+b)^{-1}, \tag{17}$$

where $a=\frac{2k^2+4k+\omega k^2+6}{2k+4}$ and b is an arbitrary consant of integration, where $a>0$.

Since $u=kv$

$$\therefore u=k(at+b)^{-1}. \tag{18}$$

By using equations (17), (18) we get

$$A=(at+b)^{\frac{1}{a}}, \tag{19}$$

$$\phi=(at+b)^{\frac{k}{a}}. \tag{20}$$

From (14), we get

$$C=(at+b)^{1-(\frac{k+2}{a})}.$$

We have

$$B=A,$$

$$\therefore B=(at+b)^{\frac{1}{a}}. \tag{21}$$

The metric (1) reduces to

$$ds^2=dt^2-(at+b)^{2/a}dx^2-(at+b)^{2/a}e^{-2\alpha x}dy^2-(at+b)^{2-2(\frac{k+2}{a})}dz^2, \quad a>0. \tag{22}$$

The metric (22) corresponds to a homogeneous but anisotropic space whose total volume increases with increasing t. The model has singularity at $t=-\frac{b}{a}$.

However, the space time (22) can be written in the form

$$ds^2=dt^2-t^{\frac{2}{a}}dx^2-t^{\frac{2}{a}}e^{-2\alpha x}dy^2-t^{\frac{2(a-k-2)}{a}}dz^2, \quad a>0 \tag{23}$$

(using a shift of origin of t and by a scale transformation).

3 Physical Properties

The spatial volume of the model is given by

$$V = (-g)^{1/2} = t^{\frac{a-k}{a}} e^{-\alpha x}, \quad a > 0. \tag{24}$$

The expansion scalar

$$\theta = \frac{1}{3} u^i_{;i},$$

$$\theta = \frac{1}{3} \frac{(a-k)}{at}, \quad a > 0. \tag{25}$$

The non vanishing component of shear tensor (σ_{ij}) are given by

$$\sigma_{11} = \frac{(a-k)t^{(\frac{2-a}{a})}}{9a}, \tag{26}$$

$$\sigma_{22} = \frac{(a-k)t^{\frac{2-a}{a}}}{9a} e^{-2\alpha x}, \tag{27}$$

$$\sigma_{33} = \frac{(a-k)t^{(\frac{a-2k-4}{a})}}{9a}, \tag{28}$$

$$\sigma_{44} = \frac{-2(a-k)}{9at}. \tag{29}$$

Shear scalar

$$\sigma^2 = \frac{7(a-k)^2}{162a^2t^2}. \tag{30}$$

The rate of expansion (H_i) along X, Y, Z axes are

$$H_1 = H_2 = \frac{2}{at} \tag{31}$$

and

$$H_3 = \frac{2(a-k-2)}{t}. \tag{32}$$

Deceleration parameter (q) is given by Finstein et al. [8]

$$q = -3\theta^{-2}\left[\theta_1 \alpha u^\alpha + \frac{1}{3}\theta^2\right],$$

which in this case has the expression

$$q = \frac{8a+k}{a-k}. \tag{33}$$

Here

$$\lim_{t\to\infty} \frac{\sigma}{\theta} = \sqrt{\frac{7}{2}} \neq 0. \tag{34}$$

4 Discussion

Since $\lim_{t\to\infty}\frac{\sigma}{\theta}\neq 0$, the model does not approach isotropy for large values of t.

From (24), (25) and (30) the spatial volume tends to infinity and the expansion scalar θ and shear scalar σ tends to zero as $t\to\infty$.

From (31) and (32) the rate of expansion of the model is a function of t. The model starts with a big-bang at $t=0$. The expansion in the model decreases as t increases and the expansion in the model stops at $t=\infty$.

The deceleration parameter will act as an indicator of the existence of inflation.

From (33) we get $q>0$. Here the positive value of q shows that the model (23) decelerate in the standard way.

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